

# Einstein manifolds with skew torsion

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## Abstract

Abstract. This paper is devoted to the first systematic investigation of manifolds that are Einstein for a connection  $\nabla$  with skew symmetric torsion. We derive the Einstein equation from a variational principle and prove that, for parallel torsion, any Einstein manifold with skew torsion has constant scalar curvature; and if it is complete of positive scalar  $\nabla$ -curvature, it is necessarily compact and it has finite first fundamental group  $\pi_1$ . The longest part of the paper is devoted to the systematic construction of large families of examples. We discuss when a Riemannian Einstein manifold can be Einstein with skew torsion. We give examples of almost Hermitian, almost metric contact, and  $G_2$  manifolds that are Einstein with skew torsion. For example, we prove that any Einstein-Sasaki manifold and any 7-dimensional 3-Sasakian manifolds admit deformations into an Einstein metric with parallel skew torsion.

## 1 Preliminaries

### 1.1 Introduction

Torsion, and in particular skew torsion, has been a topic of interest to both mathematicians and physicists in recent decades. The first attempts to modify general relativity by introducing torsion go back to the 1920's with the work of É. Cartan [Car23], and were deepened—in modified form—from the 1970's in Einstein-Cartan theory. More recently, the torsion of a connection makes its appearance in superstring compactifications, where the basic model of type II string theory consists of a Riemannian manifold, a connection with skew torsion, a spinorial field and a dilaton function.

From the mathematical point of view, skew torsion has played a significant role in the proof of the local index theorem for Hermitian non-Kähler manifolds [Bis89] and it is a standard tool for the investigation of non-symmetric homogeneous spaces, since the canonical connection of such a space does not coincide with the Levi-Civita connection anymore [TV83]. In generalized geometry [Hit10, Gua03], there are natural connections with skew torsion, the exterior derivative of the  $B$ -field.

Torsion is also ubiquitous in the theory of non-integrable geometries. This field has been revived in recent years through the development of superstring theory. Firstly, integrable geometries (like Calabi-Yau manifolds, Joyce manifolds, etc.) are exact solutions of the Strominger model with vanishing  $B$ -field. By deforming these vacuum equations and looking for models with non-trivial  $B$ -field, a new mathematical approach implies that solutions can be constructed geometrically from non-integrable geometries with torsion (for example, almost Hermitian, almost metric contact or weak  $G_2$  structures). If  $(M, g)$  denotes a Riemannian manifold, we will write any metric connection on  $M$  as  $(\nabla^g$  denotes the Levi-Civita connection)

$$\nabla_X Y = \nabla_X^g Y + A(X, Y).$$

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We say that  $\nabla$  has skew torsion if the contraction of its torsion  $H(X, Y)$  with  $g$ ,  $H(X, Y, Z) := g(H(X, Y), Z)$ , is totally antisymmetric. In this case,  $A(X, Y) = 1/2H(X, Y, -)$ . From the three Cartan classes of torsion, this is by far the richest and the best understood: such connections are always complete, they are the only ones with non-trivial coupling to the Dirac operator such that the resulting Dirac operator is still formally self-adjoint [Fri79], and many non-integrable geometric structures admit a *unique* invariant connection with totally antisymmetric torsion, thus it is a natural replacement for the Levi-Civita connection [FI02]. Also, many new results on the holonomy properties of connections with skew torsion are now available [OR12], and a lot of effort has been devoted by many researchers to the construction of geometrically interesting examples.

**Outline.** In this article, we propose a notion of ‘Einstein manifold with skew torsion’ for an  $n$ -dimensional Riemannian manifold equipped with a metric connection with skew torsion<sup>1</sup>. Our approach will be mostly dimension independent and deal, where possible, with general issues; the comparison with the results obtained previously in General Relativity (Section 1.2) will illustrate how our approach differs from the previous work in the area. We start by deducing the Einstein equation with skew torsion from a variational principle. In order to investigate the curvature properties of  $\nabla$ -Einstein manifolds, more assumptions are needed; for example, easy examples illustrate that, in general, the scalar curvature will not be constant. We show that a very suitable restriction is to impose that the torsion be parallel,  $\nabla H = 0$ . There are several families of manifolds that are classically known to admit parallel characteristic torsion, namely nearly Kähler manifolds, Sasakian manifolds, nearly parallel  $G_2$ -manifolds, and naturally reductive spaces; these classes have been considerably enlarged in more recent work (see [Vai79], [GO98], [AFS05], [Ale06], [Fri07], [Sch07]), leading to a host of instances to which our theorems can be applied. The key result illustrating that this is the ‘right’ condition is the following: If  $\nabla H = 0$ , any  $\nabla$ -Einstein manifold has constant scalar curvature (both Riemannian and of the connection with torsion); and if it is complete of positive scalar  $\nabla$ -curvature, it is necessarily compact and it has finite first fundamental group  $\pi_1$ . Thus, we obtain the best possible analogy to the Riemannian case. We then discuss an easy, but powerful criterion when a Riemannian Einstein space will be  $\nabla$ -Einstein for a given torsion 3-form.

The longest part of the paper is devoted to the systematic construction of examples in different situations. We first treat the case  $n = 4$ , where the second author had proposed an alternative definition of ‘Einstein with torsion’ based on the phenomenon of self-duality [Fer10, Fer11]. In general, this is a different concept, but we will show that they coincide if one assumes parallel torsion. Under this condition, we observe that a 4-dimensional Hermitian Einstein manifold is locally isometric to  $\mathbb{R} \times S^3$ . After a quick discussion of the Lie group case, we treat almost Hermitian manifolds in dimension 6, where we identify a class of homogeneous manifolds of type  $W_1 \oplus W_3$  that is always Einstein with parallel skew torsion; this includes, in particular, all nearly Kähler manifolds. For almost contact manifolds,  $\nabla$ -Einstein implies  $\nabla$ -Ricci-flatness, since the contact distribution is a  $\nabla$ -parallel vector field. We prove that *every* Einstein-Sasaki manifold with its characteristic torsion admits a deformation into a  $\nabla$ -Einstein-Sasaki manifold. Thus, there is a multitude of  $\nabla$ -Ricci-flat Einstein spaces in all odd dimensions. The 7-dimensional case is treated separately because of its relevance for  $G_2$  geometry. Again, all nearly parallel  $G_2$  manifolds are Einstein with parallel skew torsion; moreover, we show that any 7-dimensional 3-Saskian manifold carries three different connections that turns it into an Einstein manifold with parallel skew torsion, and that it admits a deformation of the metric that carries again an Einstein structure with parallel skew torsion. Finally, we present several  $\nabla$ -Einstein structures on Aloff-Wallach manifolds  $SU(3)/S^1$ ; several of them are new, i.e. not among those that were predicted theoretically in the previous sections.

We end this outline with some conjectural remarks. In the past years, there has been a revived interest in higher dimensional black holes, i.e. Ricci flat manifolds with Lorentzian signature, because of the exciting discovery of new horizon topologies (‘black rings’) and the option to use

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<sup>1</sup> To prevent any confusion: This is to be understood as a generalization of the *mathematical* Einstein equation, and not as an alternative field equation for the gravitational field.

these as more sophisticated backgrounds for superstring theories. On the other side, the use of skew torsion is by now a well-established tool in superstring compactifications. Thus, we believe that Einstein spaces with torsion will be of interest for future developments in this area as well, although the present paper will not deal with these issues.

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## 1.2 Einstein spaces with torsion in General Relativity

The first attempts to introduce torsion as an additional 'data' for describing physics in general relativity go back to Cartan himself [Car24]. Viewing torsion as some intrinsic angular momentum, he derived a set of gravitational field equations from a variational principle, but postulated that the energy-momentum tensor should still be divergence-free, a condition too restrictive for making this approach useful. The idea was taken up again in broader context in the late fifties. The variation of the scalar curvature and of an additional Lagrangian generating the energy-momentum and the spin tensors on a space-time endowed with a metric connection with torsion yielded the two fundamental equations of *Einstein-Cartan theory*, first formulated by Kibble [Kib61] and Sciama (see his article in [Inf62]). The first equation can, by some elimination process, be reduced to an equation which is similar to Einstein's classical field equation of general relativity with an effective energy momentum tensor  $T_{\text{eff}}$  depending on torsion, the second one relates the torsion to the spin density (in the absence of spin, the torsion vanishes and the whole theory reduces to Einstein's original formulation of general relativity). A. Trautman provided an elegant formulation of Einstein-Cartan theory in the language of principal fibre bundles [Tra73]. For a general review of gravity with spin and torsion including extensive references, we recommend the article [HHKN76] or the new 'source' book [BH12], which contains most of the articles cited in this section with extensive commentaries.

For this article, our main interest will be in general results about and exact solutions of the Einstein equation with torsion. Recall that according to Cartan [Car25], the torsion of a metric connection  $\nabla$  on  $(M, g)$  is either in  $\Lambda^3(M)$ ,  $TM$ , or a  $\frac{n(n^2-4)}{3}$ -dimensional representation space. In dimension four, these components are called in the physical literature the axial vector (since  $\Lambda^3(M^4) \cong TM^4$ ), the vector, and the tensor part of torsion. As will be discussed in Section 2.3.1, requiring the torsion to be skew symmetric (i.e., only 'axial' torsion) is rather restrictive in dimension 4, in particular if one imposes further mathematical conditions like parallel torsion. Thus, most models of general relativity with torsion allow a priori all three possibilities. Only few exact solutions to the Einstein equations with torsion appear in the literature; these are mainly of two types,

- (1) Generalizations of classical solutions: On the Schwarzschild solution  $(\mathbb{R} \times S^3, g_S)$ , one can construct a metric  $\nabla$ -Ricci flat connection with torsion, but it is of mixed torsion type [OMBH97], see also Example 2.15; with a rotationally symmetric Ansatz for metric and torsion, one obtains solutions of Schwarzschild-De Sitter [Bae81] or Kerr-type [MBG87], again of mixed torsion.
- (2) Conformal changes of the flat Minkowski metric  $(\mathbb{R}^4, g_M)$ : for example, the following Ansatz

for metric and torsion

$$\tilde{g} = e^{2\omega(t)}(-dt^2 + dx^2 + dy^2 + dz^2), \quad H = f(t) dx \wedge dy \wedge dz$$

can be adjusted in such a way to yield an exact solution of both field equations, this time with pure axial torsion [Len84].

As far as we know, no general investigation of Einstein manifolds with torsion was carried out. In practice, torsion turned out to be hard to detect experimentally, since all tests of general relativity are based on experiments in empty space. Many concepts that Einstein-Cartan theory inspired are still of relevance (see [HMMN95] for a generalization with additional currents and shear, [Tra99] for optical aspects, [RT03] for the link to the classical theory of defects in elastic media). In cosmology, Einstein-Cartan theory is again being considered in recent times (see for example [Pop11]).

### 1.3 Notations and review of curvature relations

We end this Section by recalling a few standard identities. Let  $(M, g)$  be a Riemannian manifold and  $H \in \Lambda^3(M)$ . The metric connection with skew torsion  $H$  is

$$\nabla_X Y = \nabla_X^g Y + \frac{1}{2}H(X, Y, -).$$

Quantities referring to the Levi-Civita connection will carry an upper index  $g$ , while quantities associated with the new connection will have an upper index  $\nabla$ . For example,  $s^g$  and  $s^\nabla$  will be the Riemannian and the  $\nabla$ -scalar curvatures, respectively. From the 3-form  $H$ , we can define an associated algebraic 4-form  $\sigma_H$ , quadratic in  $H$ , given by

$$2\sigma_H = \sum_{i=1}^n (e_i \lrcorner H) \wedge (e_i \lrcorner H)$$

where  $e_1, e_2, \dots, e_n$  denotes an orthonormal frame of  $TM$ . The following well-known curvature identities are crucial for the topic of this paper; they can for example be found in [FI02], [Agr06]. We introduce the tensor

$$S(X, Y) := \sum_{i=1}^n g(H(e_i, X), H(e_i, Y)) = \sum_{i,j=1}^n H(e_i, X, e_j) H(e_i, Y, e_j) \quad (1.1)$$

that measures the (symmetric part of the) difference between the Riemannian and the  $\nabla$ -curvature. We normalize the length of a 3-form  $H$  as  $\|H\|^2 = \frac{1}{6} \sum_{i,j} g(H(e_i, e_j), H(e_i, e_j))$ .

**Theorem 1.1.** *The Riemannian curvature quantities and the  $\nabla$ -curvature quantities are related by*

$$\begin{aligned} R^\nabla(X, Y, Z, W) &= R^g(X, Y, Z, W) + \frac{1}{4}g(H(X, Y), H(Z, W)) + \frac{1}{4}\sigma_H(X, Y, Z, W) \\ &\quad + \frac{1}{2}\nabla_X H(Y, Z, W) - \frac{1}{2}\nabla_Y H(X, Z, W) \\ \text{Ric}^\nabla(X, Y) &= \text{Ric}^g(X, Y) - \frac{1}{4}S(X, Y) - \frac{1}{2}\delta H(X, Y) \\ s^\nabla &= s^g - \frac{3}{2}\|H\|^2 \end{aligned}$$

## 2 Einstein metrics with skew torsion

### 2.1 The variational principle

The standard Einstein equations of Riemannian geometry can be obtained by a variational argument. They are the critical points of the Hilbert functional

$$g \longmapsto \int_M [s^g - 2\Lambda] \, \text{dvol}_g,$$

where  $\Lambda$  is a cosmological constant. Thus, one way of obtaining Einstein equations with skew torsion is to look for the critical points (with respect to the metric) of the following functional

$$(g, H) \mapsto \int_M [s^\nabla - 2\Lambda] \, \text{dvol}_g = \int_M \left[ s^g - \frac{3}{2} \|H\|^2 - 2\Lambda \right] \, \text{dvol}_g.$$

For this, we will study the variation of  $\|H\|_g^2$  with respect to  $g$ ; the torsion  $H$  does *not* yet need to be  $\nabla$ -parallel.

**Theorem 2.1.** *The critical points of the functional*

$$\mathcal{L}(g, H) = \int_M [s^\nabla - 2\Lambda] \, \text{dvol}_g$$

*are given by pairs  $(g, H)$  satisfying the equation*

$$-\text{Ric}^\nabla + \frac{1}{2} s^\nabla g - \Lambda g = 0.$$

*In particular, the antisymmetric part of  $\text{Ric}^\nabla$  vanishes, i.e.,  $H$  is coclosed.*

*Proof* — We use the summation convention throughout the proof in order to increase readability. Set  $g(t) = g + th$  and  $\{e_i(t)\}$  an orthonormal basis for  $g(t)$  such that  $e_i(0) = e_i$ . We have, according to our normalization, the following identity

$$\|H\|_{g(t)}^2(t) = \frac{1}{6} g(t)(H(e_i(t), e_j(t)), H(e_i(t), e_j(t))).$$

Taking the derivative in order to  $t$  and making  $t = 0$ , we get

$$\begin{aligned} dt\|H\|_{g(t)}^2|_{t=0} &= \frac{1}{6} h(H(e_i, e_j), H(e_i, e_j)) + \frac{1}{3} g(dt(H(e_i(t), e_j(t))|_{t=0}, H(e_i, e_j)) \\ &= \frac{1}{6} h(g(H(e_i, e_j), e_k) e_k, g(H(e_i, e_j), e_l) e_l) + \\ &\quad + \frac{1}{3} g(H(dte_i|_{t=0}, e_j) + H(e_i, dte_j|_{t=0}, H(e_i, e_j)) \\ &= \frac{1}{6} g(H(e_i, e_j), e_k) g(H(e_i, e_j), e_l) h(e_k, e_l) + \\ &\quad + \frac{1}{3} g(H(dte_i|_{t=0}, e_j), H(e_i, e_j)) + \frac{1}{3} g(H(e_i, dte_j|_{t=0}, H(e_i, e_j)) \end{aligned}$$

Using now the new tensor field  $S$  defined by equation (1.1), we have

$$\begin{aligned} dt\|H\|_{g(t)}^2|_{t=0} &= \frac{1}{6} H(e_i, e_j, e_k) H(e_i, e_j, e_l) h(e_k, e_l) + \frac{1}{3} S(dte_i|_{t=0}, e_i) + \frac{1}{3} S(dte_j|_{t=0}, e_j) \\ &= \frac{1}{6} (S, h)_g + \frac{2}{3} S(dte_i|_{t=0}, e_i) = \frac{1}{6} (S, h)_g + \frac{2}{3} S(g(dte_i|_{t=0}, e_k) e_k, e_i) \\ &= \frac{1}{6} (S, h)_g + \frac{2}{3} S(e_k, e_i) g(dte_i|_{t=0}, e_k) \end{aligned}$$

If we differentiate the equality  $g(t)(e_i(t), e_j(t)) = \delta_{ij}$  with respect to  $t$  and replace  $t = 0$ , we obtain the following equation

$$g(dte_i|_{t=0}, e_j) + g(e_i, de_j|_{t=0}) + h(e_i, e_j) = 0.$$

Using the above identity and the fact that  $T_g^H$  is symmetric, we then get that

$$\begin{aligned} dt\|H\|_{g(t)}^2|_{t=0} &= \frac{1}{6} (S, h)_g + \frac{1}{3} T_g^H(e_k, e_i) (g(dte_i|_{t=0}, e_k) + g(e_i, dte_k|_{t=0})) \\ &= \frac{1}{6} (S, h)_g - \frac{1}{3} S(e_k, e_i) h(e_k, e_i) \\ &= \frac{1}{6} (S, h)_g - \frac{1}{3} (S, h)_g = -\frac{1}{6} (S, h)_g. \end{aligned}$$

Finally we can conclude that  $\|H\|_{g(t)}^2$  has a first order Taylor expansion as follows

$$\|H\|_{g(t)}^2 = \|H\|_g^2 - \frac{1}{6} (S, h)_g t + o(t^2).$$

We can then calculate the stationary points for the functional  $\mathcal{L}(g, H)$  by finding the solutions to the equation

$$\frac{d}{dt}\mathcal{L}(g + th, H) = 0.$$

Moving the derivative under the integral sign, this is then equivalent to

$$\int_M \frac{d}{dt} \left[ \left( s^{g(t)} - \frac{3}{2} \|H\|_{g(t)}^2 - 2\Lambda \right) \text{dvol}_{g(t)} \right]_{|t=0} = 0.$$

Recall from the classical theory that the following identities hold

$$s^g(t) = s^g + (\text{div}(X) - (\text{Ric}^g, h)_g)t + o(t^2),$$

where  $X$  is a vector field whose particular form is not important for us, and

$$\text{dvol}_{g(t)} = \text{dvol}_g + \left( \frac{1}{2} (h, g)_g \text{dvol}_g \right) t + o(t^2).$$

Then the stationary points of our functional  $\mathcal{L}$  are given by the equation

$$\int_M \left[ \text{div}(X) - (\text{Ric}^g, h)_g - \frac{3}{2} \left( -\frac{1}{6} S, h \right)_g + \frac{1}{2} \left( s^g - \frac{3}{2} \|H\|^2 - 2\Lambda \right) (g, h)_g \right] \text{dvol}_g = 0.$$

The divergence term integrates to zero and simplifying the expression we get

$$\int_M \left[ \left( -\text{Ric}^g + \frac{1}{4} S + \frac{1}{2} s^\nabla g - \Lambda g, h \right)_g \right] \text{dvol}_g = 0.$$

Noticing that  $(\cdot, \cdot)_g$  is a scalar product on the space of all symmetric 2-tensors and that  $h$  is arbitrary we can then conclude that

$$-\text{Ric}^g + \frac{1}{4} S + \frac{1}{2} s^\nabla g - \Lambda g = 0$$

which is then equivalent to having  $-\text{Ric}^\nabla + \frac{1}{2} s^\nabla g - \Lambda g = 0$ . □

**Definition 2.2.** A triple  $(M, g, H)$  is said to be ‘Einstein with skew torsion’ or just ‘ $\nabla$ -Einstein’ if the connection  $\nabla$  with torsion  $H$  satisfies the Einstein equation

$$\text{Ric}^\nabla = \frac{s^\nabla}{n} g.$$

It will be called ‘Einstein with parallel skew torsion’ if in addition it satisfies  $\nabla H = 0$ .

In particular,  $\text{Ric}^\nabla$  has to be symmetric by definition, so our definition implies that an  $\nabla$ -Einstein manifold will have coclosed 3-form,  $\delta H = 0$ .

*Example 2.3.* All manifolds admitting a flat metric connection with skew torsion will be trivially  $\nabla$ -Einstein with skew torsion. These were studied by É. Cartan and J. Schouten [CS26] who argued (with a wrong proof) that, up to universal cover, such manifolds are products of a Lie group (in the case where the torsion is parallel) or otherwise the 7-sphere (where the torsion is closed). A modern, classification-free proof using holonomy theory can be found in [AF10b].

## 2.2 Curvature properties

One of the most important features of Riemannian Einstein spaces is the fact that their scalar curvature is constant, and the many consequences that follow from it. We begin with an easy example illustrating that  $\nabla$ -Einstein manifolds with non-constant scalar curvature exist:

*Example 2.4* (An example on  $S^3$ ). Consider the 3-dimensional sphere  $S^3$  and take  $g$  to be the round metric. Then it is well known that  $(S^3, g)$  is Einstein with  $s^g = 6$  and a parallelizable manifold. Let  $f : S^3 \rightarrow \mathbb{R}$  be any non-constant smooth function (like the height function) and consider the three-form  $H$  given by  $H = 2fe^1 \wedge e^2 \wedge e^3$ . Then the connection defined by  $\nabla = \nabla^g + \frac{1}{2}H$  is Einstein with skew torsion and

$$s^\nabla = s^g - \frac{3}{2}\|H\|^2 = 6 - 6f(x)^2,$$

which is clearly not a constant; the example also shows that an Einstein manifold with skew torsion can have scalar curvature of any sign, even in the compact case.

Thus, extra conditions are required to conclude the constancy of  $s^\nabla$ . We shall argue that a sufficient — and very natural — condition to impose is that  $\nabla H = 0$ , i.e. the torsion of the connection is parallel. As in the Riemannian situation, the second Bianchi identity is the key ingredient.

**Proposition 2.5** (Second Bianchi identity). *Let  $(M, g, H)$  be a Riemannian manifold equipped with a connection  $\nabla$  with skew torsion  $H$  such that  $\nabla H = 0$ . Then*

$${}^{WXY}_{\sigma} \nabla_W R^\nabla(X, Y)Z = {}^{WXY}_{\sigma} \left( R^g(W, H(X, Y))Z - \frac{1}{2}R^g(W, X)H(Y, Z) \right)$$

*Proof* — For the  $(1, 3)$ -curvature tensor quantities the following identity holds

$$R^\nabla(X, Y)Z = R^g(X, Y)Z + \frac{1}{2}H(H(X, Y), Z) + \frac{1}{4}H(H(Y, Z), X) + \frac{1}{4}H(H(Z, X), Y)$$

Set

$$R^H(X, Y)Z = \frac{1}{2}H(H(X, Y), Z) + \frac{1}{4}H(H(Y, Z), X) + \frac{1}{4}H(H(Z, X), Y)$$

and notice that since  $\nabla H = 0$  then also  $\nabla R^H = 0$ . Then it is easy to check that

$$\nabla_W R^\nabla(X, Y)Z = \nabla^g R^g(X, Y)Z - \frac{1}{2}(R^g(H(W, X), Y)Z + R^g(X, H(W, Y))Z + R^g(X, Y)H(W, Z))$$

Now we just need to take the cyclic permutation, use the second Bianchi identity for  $R^g$  and the proposition follows.  $\square$

**Corollary 2.6.** *Assume  $\nabla H = 0$ . The divergence of the  $\nabla$ -Ricci tensor is proportional to the derivative of the  $\nabla$ -scalar curvature, more precisely:*

$$\delta \text{Ric}^\nabla + \frac{1}{2}ds^\nabla = 0.$$

*1st proof* — Taking traces in the second Bianchi identity for  $R^\nabla$  we immediately get the following equation

$$\begin{aligned} -\nabla_X R^\nabla(e_i, e_j, e_j, e_i) + \nabla_{e_i} R^\nabla(e_j, X, e_i, e_j) + \nabla_{e_j} R^\nabla(e_i, X, e_j, e_i) = \\ = R^g(X, H(e_i, e_j), e_i, e_j) + R^g(e_j, H(X, e_i), e_i, e_j) + R^g(e_i, H(e_j, X), e_i, e_j) \\ - \frac{1}{2}R^g(X, e_i, H(e_j, e_i), e_j) - \frac{1}{2}R^g(e_i, e_j, H(X, e_i), e_j) \end{aligned}$$

which then simplifies to

$$\begin{aligned} -ds^\nabla(X) - 2\delta^\nabla \text{Ric}^\nabla(X) = \\ = R^g(X, H(e_i, e_j), e_i, e_j) + \frac{1}{2}R^g(X, e_i, H(e_i, e_j), e_j) + \frac{5}{2}\text{Ric}^g(H(X, e_i), e_i) \\ = H(e_i, e_j, e_k)R^g(X, e_k, e_i, e_j) + \frac{1}{2}H(e_i, e_j, e_k)R^g(X, e_i, e_k, e_j) + \frac{5}{2}H(X, e_i, e_k)\text{Ric}^g(e_k, e_i) \end{aligned}$$

Now the first two terms on the right-hand-side simplify because  $H$  is antisymmetric and  $R^g$  satisfies the first Bianchi identity and the last term vanishes because  $H$  is antisymmetric and  $\text{Ric}^g$  is symmetric. Finally, observe that since  $H$  is totally antisymmetric, the  $\nabla$ -divergence of any symmetric  $(0, 2)$ -tensor is the same as the usual divergence ([Agr06], Proposition A.2), hence  $\delta^\nabla \text{Ric}^\nabla = \delta \text{Ric}^\nabla$ .  $\square$

*2nd proof* — Let  $S$  be the symmetric 2-tensor introduced before that satisfies  $\text{Ric}^\nabla = \text{Ric}^g - S/4$ . Notice that  $S$  is  $\nabla$  parallel since it is a composition of two parallel tensors. Then

$$\delta^\nabla \text{Ric}^\nabla = \delta^\nabla \text{Ric}^g,$$

and this is again  $\delta \text{Ric}^g$  by the preceding comment on divergences. But from the Riemannian case, we know that  $\delta^g \text{Ric}^g = -\frac{1}{2} ds^g$ . Since  $H$  is parallel,  $d(\|H\|^2) = 0$ , hence the relation between the  $\nabla$ - and the Riemannian scalar curvature implies  $ds^g = ds^\nabla$ .  $\square$

Using the corollary of the second Bianchi identity we can prove the following.

**Proposition 2.7.** *Assume  $\nabla H = 0$ . If  $\delta \text{Ric}^g = 0$ , the scalar curvatures  $s^\nabla$  and  $s^g$  are constant. In particular, this holds if  $(M, g, H)$  is Einstein with parallel skew torsion  $H$ .*

*Proof* — The first claim is immediate. For the  $\nabla$ -Einstein case, taking the divergence on both sides of the equation  $\text{Ric}^\nabla = \frac{s^\nabla}{n} g$  and using the proposition above we get that  $-\frac{1}{2} ds^\nabla = \frac{1}{n} ds^\nabla$ . Therefore  $ds^\nabla = 0$  and  $s^\nabla$  is constant. But as observed before, a parallel torsion form has constant length, hence the relation between the scalar curvatures (see Theorem 1.1) implies that the Riemannian scalar curvature will be constant as well.  $\square$

*Remark 2.8.* There are other situations with  $\nabla H = 0$  where one can conclude that the scalar curvatures  $s^\nabla$  has to be constant: For example, this holds if  $M$  is spin and if there exists a non trivial parallel spinor field  $\psi$ ,  $\nabla \psi = 0$  (see [FI02, Cor. 3.2]). Since any  $G_2$  manifolds with a characteristic connection  $\nabla$  admits a  $\nabla$ -parallel spinor (see Section 2.3.5) and many of them are known to have parallel torsion, many examples of this kind that are not Einstein with skew torsion exist.

**Corollary 2.9.** *Any complete connected Riemannian manifold  $(M, g, H)$  that is Einstein with parallel skew torsion  $H$  and with positive scalar curvature  $s^\nabla > 0$  is compact and has finite first fundamental group  $\pi_1(M)$ .*

*Proof* — The crucial point is that the assumption of the Bonnet-Myers Theorem has to hold, i. e. the inequality  $\text{Ric}^g(X, X) \geq c\|X\|^2$  for some positive constant  $c$  and all  $X \in TM$ . But this is easy:

$$\text{Ric}^g(X, X) = \text{Ric}^\nabla(X, X) + \frac{1}{4} S(X, X) = \frac{s^\nabla}{n} \|X\|^2 + \frac{1}{4} S(X, X) \geq \frac{s^\nabla}{n} \|X\|^2,$$

since  $S$  is a non-negative tensor by definition. All claims now follow from the classical Riemannian results. Observe that it is not necessary to specify further with respect to which connection completeness is meant: A metric connection with skew torsion on a Riemannian manifold  $(M, g)$  is complete if and only if the Levi-Civita connection is complete, because their geodesics coincide.  $\square$

We describe now how our notion of Einstein with parallel skew torsion is consistent with the algebraic decomposition of the curvature tensor. Let  $\mathcal{CM}$  be the space of all symmetric curvature tensors on  $TM$ . Consider the Bianchi map

$$b : \mathcal{CM} \longrightarrow \mathcal{CM}, \quad R_{abcd} \longmapsto R_{abcd} + R_{bcad} + R_{cabd}.$$

It is well known (see [Bes87] for details) that  $S^2(\Lambda^2 M) = \ker b \oplus \text{im } b$  and that  $\text{im } b = \Lambda^4 M$ . Suppose now that our Riemannian manifold with torsion  $(M, g, H)$  is such that  $H$  is  $\nabla$  parallel.



Then the Riemannian curvature tensor simplifies to

$$R^\nabla(X, Y, Z, W) = R^g(X, Y, Z, W) + \frac{1}{4}g(H(X, Y), H(Z, W)) + \frac{1}{4}\sigma_H(X, Y, Z, W).$$

Observe that in this case  $R^\nabla$  is indeed in  $S^2(\Lambda^2 M)$ . We have the following proposition:

**Proposition 2.10.** *Let  $(M, g, H)$  be such that  $\nabla H = 0$ . Then  $R^\nabla$  decomposes under the Bianchi map as  $R^\nabla = R_k^\nabla + R_i^\nabla$ , where  $R_k^\nabla$  lies in the kernel of the Bianchi map and  $R_i^\nabla$  in its image, with*

$$R_k^\nabla(X, Y, Z, W) = R^g(X, Y, Z, W) + \frac{1}{4}g(H(X, Y), H(Z, W)) - \frac{1}{12}\sigma_H(X, Y, Z, W)$$

and

$$R_i^\nabla(X, Y, Z, W) = \frac{1}{6}\sigma_H(X, Y, Z, W)$$

*Proof* — Notice that  $\frac{XYZ}{\sigma} g(H(X, Y), H(Z, W)) = \sigma_H(X, Y, Z, W)$  and  $\frac{XYZ}{\sigma} \sigma_H(X, Y, Z, W) = 3\sigma_H(X, Y, Z, W)$ . Since  $R^g$  is in  $\ker b$  then so is  $R_k^\nabla$ . Notice also that  $\sigma_H \in \Lambda^4 M$ .  $\square$

Following the classical theory we can then decompose the  $\nabla$ -curvature tensor as

$$R^\nabla = W^\nabla + \frac{1}{n-2}(Z^\nabla \boxtimes g) + \frac{s^\nabla}{n(n-1)}g \boxtimes g + \frac{1}{6}\sigma_H$$

where  $\boxtimes$  denotes the Kulkarni-Nomizu product,  $Z^\nabla = \text{Ric}^\nabla - \frac{s^\nabla}{n}g$  is the trace-free part of the  $\nabla$ -Ricci tensor and  $W^\nabla$ , which we shall call the  $\nabla$ -Weyl tensor, can be explicitly written as

$$\begin{aligned} W^\nabla(X, Y, Z, W) &= W^g(X, Y, Z, W) + \frac{1}{4}g(H(X, Y), H(Z, W)) + \sigma_H(X, Y, Z, W) \\ &\quad + \frac{1}{4(n-2)}(g(H(e_i, X), H(e_i, W))g(Y, Z) + g(H(e_i, Y), H(e_i, Z))g(X, W) \\ &\quad - g(H(e_i, X), H(e_i, Z))g(Y, W) - g(H(e_i, Y), H(e_i, W))g(X, Z)) \\ &\quad - \frac{3\|H\|^2}{2(n-1)(n-2)}(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \end{aligned}$$

Note that  $W^\nabla$  is traceless. This can be checked by direct calculation but, of course, it also follows from the general theory. We conclude that our previous definition of  $(M, g, H)$  being Einstein with skew torsion is equivalent to  $Z^\nabla = 0$ , as it should; let us emphasize that this relies again strongly on the property  $\nabla H = 0$ .

Next, we want to clarify the relation between  $\nabla$ -Einstein and Riemannian Einstein manifolds. In a given dimension, the algebraic form of the difference tensor  $S(X, Y)$  of the curvatures as defined in equation 1.1 decides whether a Riemannian Einstein metric will yield a skew Einstein structure or not.

**Definition 2.11.** *On a Riemannian manifold  $(M, g)$ , a 3-form  $H$  will be called ‘of Einstein type’ if the difference tensor  $S(X, Y) := \sum_i g(H(e_i, X), H(e_i, Y))$  is proportional to the metric  $g$ .*

**Proposition 2.12.** *Let  $(M, g)$  be a Riemannian manifold,  $H$  a 3-form written in a local orthonormal frame  $e^1, \dots, e^n$  of  $T^*M$ ,  $H = \sum_{ijk} H_{ijk} e^i \wedge e^j \wedge e^k$ . Then  $H$  is of Einstein type if it satisfies the following conditions:*

- (1) *no term of the form  $H_{ija} e^i \wedge e^j \wedge e^a + H_{ijb} e^i \wedge e^j \wedge e^b$  with  $a \neq b$  occurs;*
- (2) *if  $i$  and  $j$  are two indices in  $\{1, \dots, n\}$  then the number of occurrences of  $i$  and  $j$  in  $H$  coincides;*
- (3) *if  $\{i, j, k\}$  and  $\{a, b, c\}$  are two sets of indices then  $H_{ijk}^2 = H_{abc}^2$ .*

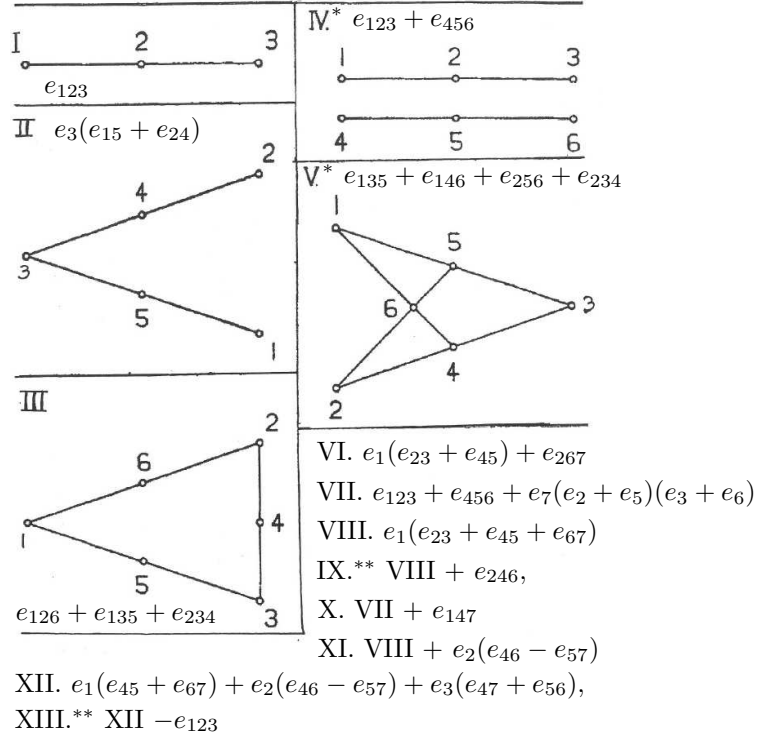


Figure 1: Normal forms of 3-forms up to dimension 7 (see Example 2.14); the picture is taken from the original article of J. Schouten from 1931 [Sch31] while for  $n = 7$  we follow [Wes81]. Any three numbers  $a, b, c$  linked by lines represent a summand  $e_a \wedge e_b \wedge e_c$ .

*Proof* — It can be checked by direct calculation that, when writing  $S$  in matrix form for the frame  $\{e_i\}$ , condition (1) guarantees  $S$  to be diagonal, while conditions (2) and (3) ensure that  $S$  is indeed a multiple of the identity matrix.  $\square$

*Remark 2.13.* Proposition 2.12 yields an easy procedure for producing further examples of  $\nabla$ -Einstein metrics with non constant scalar curvature (beyond the one given in Example 2.4) for all manifolds that are parallelizable and carry an Einstein metric (for example  $S^7$  or compact semi-simple Lie groups).

*Example 2.14.* In 1931, J. Schouten described the normal forms of 3-forms up to dimension 7 [Sch31], i. e. representatives of the  $\mathrm{GL}(n, \mathbb{R})$ -orbits for  $n \leq 7$ , see Figure 1 (the complex classification is different; see also [Wes81] for a modern account of the real classification). The normal forms marked with one resp. two stars are the representatives of generic 3-forms, i. e. the ones with dense  $\mathrm{GL}(n, \mathbb{R})$  orbit for  $n = 6$  resp.  $n = 7$  (see for example [VW12]). One checks by a direct computation that only the following 3-forms are of Einstein type in the given dimensions:

- (1) Type I. in dimension 3,
- (2) Types IV. and V. in dimension 6,
- (3) Type XIII. in dimension 7.

In particular, Riemannian Einstein manifolds  $(M, g)$  will never be  $\nabla$ -Einstein in dimensions 4 and 5. We will see later concrete examples where Proposition 2.12 can be applied to construct  $\nabla$ -Einstein metrics.

*Example 2.15.* Proposition 2.12 shows that there does not exist a three-form that will make the Schwarzschild metric  $\nabla$ -Einstein; in particular, the connection constructed in [OMBH97] has to be of mixed torsion type.

## 2.3 Examples and construction of $\nabla$ -Einstein manifolds

### 2.3.1 The case $n = 4$

In dimension 4, an alternative notion of Einstein with skew torsion was investigated by the second author in [Fer10, Fer11], based on the phenomenon of self-duality. The idea was to consider the decomposition of the curvature operator  $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$  in terms of the splitting of the bundle  $\Lambda^2 = \Lambda_+ \oplus \Lambda_-$  into self-dual and antiself-dual parts. By making the analogy with the standard definition of Einstein metric, one can set the upper-right block in this decomposition to vanish, that is, the following definition was taken. In order not to confuse it with the Einstein definition used in this article, we will call it *duality-Einstein with skew torsion*, in the sense that it is the Einstein characterisation based on (self-)duality.

**Definition 2.16.** *A triple  $(M^4, g, H)$  is said to be duality-Einstein with skew torsion if*

$$Z^\nabla + S(\nabla * H) + \frac{*dH}{4}g = 0,$$

where  $S$  denotes the symmetrization of a tensor and  $Z^\nabla$  is the symmetric trace-free part of  $\text{Ric}^\nabla$ .

This definition depends a priori on the choice of orientation, but it can be proved that for a compact manifold, this choice is irrelevant, since the metric dual of  $*H$  is a Killing vector field. Among other results, it was shown that the Hitchin-Thorpe inequality  $2\chi \geq 3|\tau|$  holds again, giving a severe topological obstruction. Since, in four dimensions,  $*H$  is a 1-form, it is not difficult to establish that the notion of duality-Einstein manifold with skew torsion is equivalent to that of Einstein-Weyl manifold, which has been a subject of intense study in the past [CP99]. Our new definition of Einstein manifolds with parallel skew torsion breaks away from Einstein-Weyl geometry for  $n \neq 4$ . This can be seen for example from the scalar curvature: while it has to be constant in our situation by Proposition 2.7, it has only constant sign in Einstein-Weyl geometry [PS93].

In general, the notions of  $\nabla$ -Einstein and duality-Einstein differ. But with the additional assumption that the torsion is parallel, they coincide: first, observe that in  $n = 4$ , condition  $\nabla H = 0$  is equivalent to  $dH = 0$  by Proposition A.1 and the fact that  $\sigma_H = 0$  in dimension four for purely algebraic reasons. Also,  $\nabla * H = 0$  as well, so the duality-Einstein equations reduces to  $Z^\nabla = 0$ , and this is our definition of an Einstein structure with parallel skew torsion. In fact, more holds: since  $*H$  is a parallel vector field, such a manifold will have to be  $\nabla$ -Ricci-flat. A result similar to the following (but under different assumptions) was proved in the second author's dissertation [Fer10].

**Theorem 2.17.** *If  $(M, g, J)$  is a 4-dimensional compact Hermitian non-Kähler manifold equipped with the Bismut connection such that it is Einstein with parallel skew torsion, then its universal cover is isometric to  $\mathbb{R} \times S^3$  with the standard product metric.*

*Proof* — The torsion of the Bismut connection  $\nabla$  is  $H = d^c\Omega$  ( $\Omega$  is the Kähler form), its Lee form  $\theta$  is the 1-form satisfying  $\theta = d^*\Omega \circ J$ . In four dimensions,  $\theta$  has the remarkable property that  $\theta = *d^c\Omega$ . So, since  $\nabla H = \nabla g = 0$ , we can conclude that  $\nabla\theta = 0$ . It is easy to see that, in four-dimensions,  $\nabla\theta = \nabla^g\theta$ . Then if  $X$  is the metric dual of  $\theta$ , we have that  $X$  is a  $\nabla^g$ -parallel vector field. If  $X = 0$  then  $d\Omega = 0$ , in which case  $(M, g, J)$  would be Kähler and  $\nabla = \nabla^g$ , a case we excluded by assumption. If  $X \neq 0$  then we have a nowhere vanishing parallel vector field. By the de Rham theorem, we have a reduction of  $TM$  under the holonomy group and that locally  $M$  is isometric to  $\mathbb{R} \times N$ , where  $N$  is a 3-dimensional manifold such that  $TN$  is the orthogonal complement of  $\{X\}$ . Since  $\text{Ric}^\nabla = 0$ , then (for details on this formula see [Fer10, Fer11])

$$\text{Ric}^g = \frac{1}{2}\|\theta\|^2g - \frac{1}{2}\theta \otimes \theta.$$

Then  $N$  is an Einstein manifold of positive Ricci curvature which implies that, since  $N$  is 3-dimensional, it is of positive sectional curvature. Finally, we can conclude that  $M$  is locally isometric to  $\mathbb{R} \times S^3$ .  $\square$

### 2.3.2 Lie groups

Classical examples of manifolds where skew torsion arises naturally are Lie groups  $G$  equipped with a bi-invariant inner product  $g$  on the corresponding Lie algebra  $\mathfrak{g}$  (see [KN69], for example). There is a one-to-one correspondence between the set of all bi-invariant connections and the space of all bilinear functions  $\alpha : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  which are ad-invariant; in particular,  $\nabla_X^t(Y) = t[X, Y]$  defines a one-parameter family of metric connections with parallel skew torsion  $(2t - 1)[X, Y]$ . The value  $t = 1/2$  corresponds to the Levi-Civita connection, while  $t = 0$  and  $t = 1$  are the flat  $\pm$ -connections introduced by Schouten and Cartan. A routine calculation shows:

**Lemma 2.18.** *Let  $G$  be a Lie group equipped with a bi-invariant metric. Consider the 1-parameter family of connections with skew torsion  $\nabla_X^t Y = t[X, Y]$ . It has Ricci curvature*

$$\text{Ric}^{\nabla^t}(X, Y) = 4(t - t^2) \text{Ric}^g(X, Y).$$

Hence, for  $t = 0, 1$ , the connection  $\nabla^t$  is  $\nabla^t$ -Einstein (it is in fact flat), and for  $t \neq 0, 1$ ,  $\nabla^t$  is Einstein if and only if  $\nabla^g$  is Einstein.

It is well known that for any semisimple Lie group, the Killing form  $K$  is negative definite and therefore  $-K$  defines a bi-invariant metric. This metric is clearly always Einstein in the usual sense. However, not all Lie groups admit Einstein metrics; this happens for example for  $G = U(1) \times SU(2)$ . If  $(G, g)$  was Einstein, the Euler characteristic of  $G$  would be given by [Bes87]

$$\chi(G) = \frac{1}{8\pi^2} \int_G \left( \frac{s^2}{24} + \|W\|^2 \right) \text{dvol}_g.$$

Since  $\chi(G) = 0$ , this would mean that both the scalar and the Weyl curvature vanish and therefore  $G$  would be flat with respect to the Levi-Civita connection, which cannot happen since its universal cover is  $\mathbb{R} \times S^3$  and not  $\mathbb{R}^4$ .

### 2.3.3 Almost Hermitian manifolds

We will now take a look at almost Hermitian manifolds. By a result of Friedrich and Ivanov [FI02], any almost Hermitian manifold of Gray-Hervella type  $W_1 \oplus W_3 \oplus W_4$  [GH80] admits a unique Hermitian connection  $\nabla$  with skew-symmetric torsion (i.e. a connection satisfying  $\nabla g = \nabla J = 0$ , see [Gau97] for a survey). The connection is called the *characteristic connection* of the almost Hermitian structure.

*Example 2.19.* By definition, a *Calabi-Yau connection with torsion* (CYT connection for short) is a Hermitian  $2n$ -dimensional manifold such that the restricted holonomy of its characteristic connection lies in  $SU(n)$ . In [GGP08], the authors investigated such connections on principal  $T^{2k}$ -bundles over compact Kähler manifolds. By [GGP08], Proposition 5 and equation (11), one sees that the existence of a CYT connection implies that the bundle has to be  $\text{Ric}^\nabla$ -flat (the connection of interest corresponds to  $t = -1$ ). By a sophisticated topological construction, the authors construct several series of examples on  $(k - 1)(S^2 \times S^4) \# k(S^3 \times S^3)$  for all  $k \geq 1$ . Examples on  $S^3 \times S^3$  were shown to exist previously in [FPS04].

*Example 2.20.* An almost Hermitian (non-Kählerian) manifold is called *nearly Kähler* if it satisfies  $(\nabla_X^g J)(X) = 0$ , these are precisely the manifolds of type  $W_1$ . Nearly Kähler manifolds were introduced and extensively studied by A. Gray [Gra70, Gra76]. The characteristic connection is then called the Gray connection. It is a non-trivial result that its torsion is  $\nabla$ -parallel, [Kir77, AFS05]. Furthermore, A. Gray showed that any 6-dimensional nearly Kähler manifold is Einstein

[Gra76]. A routine computation in the theory of nearly Kähler manifolds shows that  $S(X, Y) = \frac{4s^g}{30}g(X, Y)$  and hence  $\text{Ric}^\nabla = \frac{2s^g}{15}g$ : Any nearly Kähler manifold is Einstein with parallel skew torsion with respect to the Gray connection. This has been implicitly known in the community and was noted several times in the literature, for example, in [AFS05], [Agr06]. In [But05], it was shown that a 6-dimensional locally homogeneous nearly Kähler manifold has to be one of the following:  $S^6$ ,  $S^3 \times S^3$  or the twistor spaces for  $S^4$  and  $\mathbb{C}P^2$  ( $\mathbb{C}P^3$  and the flag manifold  $F(1, 2)$ , resp.).

N. Schoemann investigated in [Sch07] almost Hermitian 6-manifolds admitting a characteristic connection with parallel torsion. He discovered that there is a class of manifolds whose curvature tensor is the same as in the nearly Kähler case, and hence they will be Einstein with skew torsion. Let us describe them in more detail.

**Theorem 2.21** ([Sch07, Section 4.2]). *Let  $(M^6, g)$  be an almost Hermitian manifold admitting a characteristic connection  $\nabla$  with parallel torsion  $H$  such that its reduced holonomy  $\text{Hol}_0(\nabla)$  lies inside  $\text{SO}(3)$ . Then  $(M^6, g)$  is of type  $W_1 \oplus W_3$ , locally isomorphic to an isotropy irreducible homogeneous space, and the curvature transformation is of the form  $R^\nabla : \Lambda^2(\mathbb{R}^6) \rightarrow \mathfrak{so}(3)$ ,  $R^\nabla(X, Y) = \lambda \text{pr}_{\mathfrak{so}(3)}(X \wedge Y)$ ,  $\lambda \in \mathbb{R}$ .*

In particular, the nearly Kähler structure on  $S^3 \times S^3$  is of this type, and thus we can conclude without calculation that any almost Hermitian manifold with the same curvature transformation is  $\nabla$ -Einstein with parallel skew torsion. Examples are given in Section 4.5 of [Sch07] and in [AFS05]: Besides a larger class of almost Hermitian structures on  $S^3 \times S^3$  that includes the nearly Kähler case, examples can be constructed on  $\text{SL}(2, \mathbb{C})$  (viewed as a real 6-manifold,  $\text{Ric}^\nabla = -\frac{1}{3}\|H\|^2g$ ),  $\text{SU}(2) \ltimes \mathbb{R}^2$ , and a nilpotent group  $N^6$ .

### 2.3.4 Almost contact metric manifolds

We shall now investigate contact manifolds in greater detail: We will be able to construct large classes of new Einstein structures with torsion from them.

**Definition 2.22.** *We say that  $(M, g)$  carries an almost contact metric structure if it admits a  $(1,1)$ -tensor field  $\varphi$  and a vector field  $\xi$  with dual form  $\eta$  such that  $\varphi^2 = -\text{Id} + \eta \otimes \xi$  and if  $g$  is  $\varphi$ -compatible, i. e.*

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

Almost contact manifolds have two naturally associated tensors, the fundamental form  $F$  defined by  $F(X, Y) = g(X, \varphi(Y))$  and the Nijenhuis tensor  $N$  given by

$$N(X, Y) = [\varphi(X), \varphi(Y)] - \varphi([X, \varphi(Y)]) - \varphi([\varphi(X), Y]) + \varphi^2([X, Y]) + d\eta(X, Y)\xi.$$

We say that the almost contact metric manifold  $M$  is normal if  $N = 0$  and that it is contact metric if  $2F = d\eta$ . A contact metric manifold that is also normal is called a Sasaki manifold.

Again, Friedrich and Ivanov characterised the almost contact metric structures that admit a metric connection with skew torsion satisfying  $\nabla g = \nabla \eta = \nabla \varphi = 0$ : These are precisely the manifolds for which  $N$  is a 3-form and  $\xi$  is a Killing field [FI02], the connection is unique and again called the characteristic connection. A class of manifolds satisfying these conditions are the Sasaki manifolds. In this particular case, the torsion tensor simplifies to  $H = \eta \wedge d\eta$  and also we have that  $H$  is  $\nabla$ -parallel.

**Lemma 2.23.** *Let  $(M, g, \varphi, \xi, \eta)$  be an almost contact metric structure admitting a characteristic connection. If  $M$  is  $\nabla$ -Einstein, then it is  $\nabla$ -Ricci flat.*

*Proof* — This follows immediately from the fact that  $\nabla \xi = 0$  implies  $\text{Ric}^\nabla(\xi, \xi) = 0$ . □

We will now investigate the so-called *Tanno deformation* of a Sasaki metric. The following results appears implicitly in the articles of S. Tanno [Tan68] and of E. Kim and T. Friedrich [KF00], see also J. Becker-Bender's PhD thesis for a more explicit formulation and proof [BB12].

**Theorem 2.24** ([Tan68, KF00]). *Let  $(M, g, \xi, \varphi, \eta)$  be a Sasaki manifold of dimension  $2k + 1$  and  $g_t = tg + (t^2 - t)\eta \otimes \eta$ ,  $t > 0$ , a deformation of the metric in the direction of  $\xi$ . Then  $(M, g_t, \xi_t, \varphi, \eta_t)$ , where  $\xi_t = \frac{1}{t}\xi$  and  $\eta_t = t\eta$ , is also a Sasakian manifold. The Levi-Civita connection for the metric  $g_t$  is given by*

$$\nabla^{g_t} = \nabla^g + (1 - t)(\eta \otimes \varphi + \varphi \otimes \eta).$$

For the Ricci tensor we have the following expression

$$\text{Ric}^{g_t} = \text{Ric}^g + 2(1 - t)g - (1 - t)(2kt + 2k - 1)\eta \otimes \eta.$$

We will now show that if we start with an Einstein-Sasaki or an  $\eta$ -Einstein-Sasaki manifold, i. e. a Sasaki manifold whose Riemannian Ricci tensor has the form  $\text{Ric}^g = ag + (b - a)\eta \otimes \eta$  (with one extra condition), there is a parameter  $t$  in the Tanno deformation such that the characteristic connection is Einstein with skew torsion. Recall that for  $n \geq 5$ , the coefficients  $a$  and  $b$  have to be constant for an  $\eta$ -Einstein-Sasaki manifold [Bla10].

**Theorem 2.25.** *Let  $(M^{2k+1}, g, \xi, \varphi, \eta)$  be an Einstein-Sasaki manifold or an  $\eta$ -Einstein-Sasaki manifold satisfying  $2k + 1 > b - a$ . Then there exists a parameter  $t > 0$  for which the Tanno deformation  $(M, g_t, \xi_t, \varphi, \eta_t)$  equipped with its characteristic connection*

$$\nabla^t = \nabla^{g_t} + \frac{1}{2}\eta_t \wedge d\eta_t$$

is  $\nabla^t$ -Ricci flat.

*Proof* — We start by observing that if  $H^t$  is the torsion of  $\nabla^t$  then

$$H^t = \eta_t \wedge d\eta_t = t^2 d\eta \wedge \eta = t^2 H.$$

Also since  $M$  is Sasaki,  $d\eta = 2F$  where  $F$  is the fundamental 2-form, that is,  $F(X, Y) = g(X, \varphi(Y))$ . Notice that then we can write  $H$ , viewed as a  $(1, 2)$ -tensor, as  $H(X, Y) = 2F(X, Y)\xi$ . Consider an adapted orthonormal basis for  $g$ , say,  $\{a_1, \varphi(a_1), \dots, a_k, \varphi(a_k), \xi\}$ . Then the set

$$\{t^{-1/2}a_1, t^{-1/2}\varphi(a_1), \dots, t^{-1/2}a_k, t^{-1/2}\varphi(a_k), t^{-1}\xi\}$$

is an adapted orthonormal basis for the metric  $g_t$ . For notational convenience we will relabel this basis by  $\{e_i(t), i = 1, \dots, 2k + 1\}$ . We need to analyze the tensor

$$S^t(X, Y) = g_t(H^t(e_i(t), X), H^t(e_i(t), Y)).$$

Expanding the expression for  $g_t$  and  $H^t$  we have

$$S^t(X, Y) = t^5 \sum_{i=1}^{2k+1} g(H(e_i(t), X), H(e_i(t), Y) + 4(1 - t)\eta(H(e_i(t), X))\eta(H(e_i(t), Y)))$$

and using the fact that  $H(X, Y) = 2g(X, \varphi(Y))\xi$ , this simplifies to

$$S^t(X, Y) = 4t^6 \sum_{i=1}^{2k+1} g(e_i(t), \varphi(X))g(e_i(t), \varphi(Y)).$$

Given the particular expression of our adapted basis, and recalling that  $\varphi(\xi) = 0$ , we can yet write  $S^t$  as

$$S^t(X, Y) = 4t^5 \sum_{i=1}^{2k} g(e_i, \varphi(X))g(e_i, \varphi(Y)).$$

It is easy to check that  $S^t(a_i, a_i) = S^t(\varphi(a_i), \varphi(a_i)) = 4t^5$ , for  $i = 1, \dots, k$  and that all other terms vanish. Then  $S^t$  can be written in matrix form as  $S^t = 4t^5 \text{diag}(1, \dots, 1, 0)$ . Observe also

that  $\eta \otimes \eta$  is given in this basis by  $\text{diag}(0, \dots, 0, 1)$ . Hence, the Ricci tensor is given by the expression

$$\text{Ric}^{\nabla^t} = \text{Ric}^g + 2(1-t)g - (1-t)(2kt + 2k + 1)\eta \otimes \eta - \frac{1}{4}S^t.$$

If  $(M, g)$  is Einstein, then  $\nabla^t$  will be Einstein if and only if the matrix

$$-t^5 \text{diag}(1, \dots, 1, 0) - (1-t)(2kt + 2k + 1) \text{diag}(0, \dots, 0, 1)$$

is a multiple of the identity. This happens if and only if

$$f_1(t) := t^5 - (1-t)(2kt + 2k + 1) = 0.$$

Using the Cauchy-Bolzano theorem we can conclude that such a solution exists for some  $t > 0$  (in fact, one checks numerically that for  $k = 2, \dots, 10$ , the unique positive solution lies in the interval  $[9/10, 1]$ ). Thus  $\nabla^t$  is Einstein.

In case  $(M, g)$  is only  $\eta$ -Einstein, one deduces similarly that one needs to solve the equation

$$f_2(t) := t^5 - (1-t)(2kt + 2k + 1) + (b - a) = 0.$$

The assumption  $2k + 1 > b - a$  then implies that  $f(0) < 0$ , hence the Cauchy-Bolzano theorem guarantees again the existence of a positive solution (for other values of  $b$  and  $a$ , a more detailed investigation of the equation  $f(t) = 0$  may still yield solutions  $t > 0$ , of course). In both cases, the Ricci-flatness now follows from Proposition 2.23.  $\square$

*Remark 2.26.* Notice that  $t = 1$  never gives a solution of  $f_1(t) = 0$  that is, if  $(M, g, \varphi, \xi, \eta)$  is Einstein then it is never  $\nabla$ -Einstein with respect to its characteristic connection, in accordance with Proposition 2.12 and the ensuing discussion of normal forms in dimensions 5 and 7.

*Remark 2.27.* Observe that Theorem 2.25 leads to many examples of homogeneous  $\nabla$ -Ricci flat manifolds which are not flat, as opposed to the Riemannian case (for the standard case refer, for example, to [Bes87]).

We do not know of a similar general result for constructing explicitly  $\text{Ric}^{\nabla}$ -flat manifolds in *even* dimensions. Nevertheless, such manifolds exist, see Example 2.19.

### 2.3.5 $G_2$ manifolds with torsion

We now consider the class of 7-dimensional Riemannian manifolds equipped with a  $G_2$  structure. A  $G_2$  structure can be seen as a triple  $(M, g, \omega)$  consisting of a 7-dimensional manifold, a Riemannian metric, and a 3-form of general type at any point. A  $G_2$   $T$ -manifold —  $G_2$  manifold with (skew) torsion — is a manifold equipped with a  $G_2$  structure such that there exists a one-form  $\theta$  such that  $d * \omega = \theta \wedge * \omega$ ; equivalently, these are the manifolds of Fernandez-Gray type  $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ , see [FG82]. It admits a unique connection with totally skew symmetric torsion which preserves both the metric  $g$  and the 3-form  $\omega$ , called again the characteristic connection (of the  $G_2$  structure), see [FI02].

*Example 2.28.* A  $G_2$  manifold  $(M, g, \omega)$  is called *nearly parallel*  $G_2$  if  $d\omega = \lambda * \omega$ , for some  $0 \neq \lambda \in \mathbb{R}$ . They coincide with the  $G_2$  manifolds of type  $\mathcal{X}_1$ . It is a well known fact that nearly parallel  $G_2$  manifolds are Einstein<sup>2</sup>. Furthermore, a nearly parallel  $G_2$  manifold is also Einstein with parallel skew torsion,  $\text{Ric}^{\nabla} = \frac{\lambda^2}{3}g$ . This can be easily deduced from the formulas in [FI02, p. 318] or from the spinorial argument in [Agr06], but it is also an immediate consequence of Proposition 2.12.

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<sup>2</sup> This can be found in [FKMS97]; the result is also implicitly contained in [BFGK91, Thm. 13, p.120], since one checks that the assumptions of the theorem are exactly describing the existence of a nearly parallel  $G_2$  structure. The result then follows from the fact that every spin manifold with a Killing spinor is Einstein, [Fri80]. In that time, it was just not yet fashionable to call  $G_2$  structures by this name.

*Example 2.29.* There are 7-dimensional cocalibrated  $G_2$  manifolds  $(M^7, g, \omega^3)$  with characteristic torsion  $T$  such that

$$\nabla T = 0, \quad dT = 0, \quad \delta T = 0, \quad \text{Ric}^\nabla = 0, \quad \mathfrak{hol}(\nabla) \subset \mathfrak{u}(2) \subset \mathfrak{g}_2.$$

These  $G_2$  manifolds have been described in [Fri07, Thm 5.2] (the degenerate case where  $2a + c = 0$ ).  $M^7$  is the product  $X^4 \times S^3$ , where  $X^4$  is any Ricci-flat Kähler manifold and  $S^3$  the round sphere. There is an underlying  $S^1$ -fibration, but it does not induce a contact structure (the 1-form  $\eta \simeq e_7$  describing the fibre satisfies  $d\eta \wedge d\eta = 0$ , which cannot be); hence, the example is not covered by the Tanno deformations described in Theorem 2.25.

3-Sasakian manifolds  $M$  are Riemannian Einstein spaces of positive scalar curvature carrying three compatible orthogonal Sasakian structures  $(\xi_\alpha, \eta_\alpha, \varphi_\alpha)$ ,  $\alpha = 1, 2, 3$  [IK73, BG08]. The 7-dimensional case is of particular interest because of its relation to spin geometry and  $G_2$  structures. The simply connected homogeneous 3-Sasakian 7-manifolds were classified up to isometry in [BGM94] and turn out to be  $\text{Sp}(2)/\text{Sp}(1) \cong S^7$  and  $\text{SU}(3)/S^1$ ; by [FK90], these are precisely the compact simply connected 7-dimensional spin manifolds with regular 3-Sasaki structure. Many non homogeneous examples are known [BG08].

**Theorem 2.30.** *Every 7-dimensional 3-Sasakian manifold carries three different connections that turn it into an Einstein manifold with parallel skew torsion; furthermore, it admits a deformation of the metric that carries again an Einstein structure with parallel skew torsion.*

*Proof* — It is well-known that every 7-dimensional 3-Sasakian manifold admits three linearly independent Killing spinors  $\psi_1, \psi_2, \psi_3$  [FK90]; hence, each of these Killing spinors defines a nearly parallel  $G_2$  structure  $\omega_i$  with a characteristic connection  $\nabla^i$  which is Einstein with parallel skew torsion by the previous example. As described in [AF10a, Thm. 6.2], these three connections are truly different, and have the same constant  $\lambda = -4$ , hence they satisfy  $\text{Ric}^\nabla = \frac{16}{3}g$ .

Furthermore, any 7-dimensional 3-Sasakian metric can be deformed into a  $G_2$ -structure in the following way (see [FKMS97, Fri07, AF10a] for more details). The vertical subbundle  $T^v \subset TM$  is spanned by  $\xi_1, \xi_2, \xi_3$ , its orthogonal complement is the horizontal subbundle  $T^h$ . Fix a positive parameter  $s > 0$  and consider a new Riemannian metric  $g^s$  defined by

$$g^s(X, Y) := g(X, Y) \quad \text{if } X, Y \in T^h, \quad g^s(X, Y) := s^2 \cdot g(X, Y) \quad \text{if } X, Y \in T^v.$$

We rescale the 3-forms

$$F_1 = \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{and} \quad F_2 = \frac{1}{2}(\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3) + 3\eta_1 \wedge \eta_2 \wedge \eta_3$$

to obtain the new forms

$$F_1^s := s^3 F_1, \quad F_2^s := s F_2, \quad \omega^s := F_1^s + F_2^s.$$

One shows that  $(M^7, g^s, \omega^s)$  is a Riemannian 7-manifold equipped with a cocalibrated  $G_2$ -structure  $\omega^s$ , hence it admits a characteristic connection  $\nabla$  with skew torsion

$$H_s = \left[ \frac{2}{s} - 10s \right] (s\eta_1) \wedge (s\eta_2) \wedge (s\eta_3) + 2s\omega^s.$$

The Ricci tensor of the characteristic connection of  $\nabla$  (written as an endomorphism) is given by the formula [Fri07]

$$\text{Ric}^\nabla = 12(1 - s^2) \text{Id}_{T^h} \oplus 16(1 - 2s^2) \text{Id}_{T^v}.$$

If  $s = 1$  (the 3-Sasakian case), then  $\text{Ric}^\nabla$  vanishes on the subbundle  $T^h$ . For  $s = 1/\sqrt{5}$ , the Ricci tensor is proportional to the metric,  $\text{Ric}^\nabla = (48/5) \text{Id}_{TM^7}$ , so we obtain an Einstein structure with skew torsion as claimed. As already observed in [FKMS97, Thm. 5.4], it is nearly parallel  $G_2$ , so it has parallel torsion. Finally, it was shown in [AF10a, Cor. 7.1] that its corresponding Killing spinor is  $\psi_0$ , the *canonical spinor* introduced in the same paper. Since it is related to the Killing spinors of the underlying 3-Sasakian structure by  $\psi_i = \xi_i \cdot \psi_0$  [AF10a, Thm. 6.1], we see that we obtained yet another different nearly parallel  $G_2$  structure on the 3-Sasaki manifold we started with. □



### 2.3.6 The Aloff-Wallach manifold $N(1, 1)$

This is a computer-aided systematic search for metric connections with skew torsion on the Aloff-Wallach manifold  $SU(3)/S^1$ . It was our goal to test how rare or common  $\nabla$ -Einstein structures are, apart from the ones that we can predict theoretically. The main result is that, indeed, additional  $\nabla$ -Einstein structures exist.

We use the computations available in [BFGK91, p.109 ff] and [AF04, p.733 ff], which we then shall not reproduce here. The manifold  $N(1, 1)$  is the homogeneous space  $SU(3)/S^1$  where we are taking the embedding  $S^1 \rightarrow SU(3)$  given by  $e^{i\theta} \mapsto \text{diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta})$ . The Lie algebra  $\mathfrak{su}(3)$  splits into  $\mathfrak{su}(3) = \mathfrak{m} + \mathbb{R}$ , where  $\mathbb{R}$  is the Lie algebra of  $S^1$  given by the considered embedding. The space  $\mathfrak{m}$  splits into  $\mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ , where all  $\mathfrak{m}_i$  are pairwise orthogonal with respect to the (negative of the) Killing form  $B(X, Y) := -\text{Re}(\text{tr } XY)/2$  of  $\mathfrak{su}(3)$ . The subspace  $\mathfrak{m}_0$  is spanned by the matrix  $L := \text{diag}(3i, -3i, 0)$ . Let  $E_{ij}$  ( $i < j$ ) be the matrix with 1 at the place  $(i, j)$  and zero elsewhere, and define  $A_{ij} = E_{ij} - E_{ji}$ ,  $\tilde{A}_{ij} = i(E_{ij} + E_{ji})$ . Then  $\mathfrak{m}_1 := \text{Lin}\{A_{12}, \tilde{A}_{12}\}$ ,  $\mathfrak{m}_2 := \text{Lin}\{A_{13}, \tilde{A}_{13}\}$  and  $\mathfrak{m}_3 := \text{Lin}\{A_{23}, \tilde{A}_{23}\}$ . We consider the two-parameter family of metrics defined by the formula

$$g_{s,y} := \frac{1}{s^2}B|_{\mathfrak{m}_0} + B|_{\mathfrak{m}_1} + \frac{1}{y}B|_{\mathfrak{m}_2} + \frac{1}{y}B|_{\mathfrak{m}_3}.$$

This is a subfamily of the family considered in [BFGK91, p.109 ff]. The isotropy representation  $\text{Ad}(\theta)$  leaves the vectors in  $\mathfrak{m}_0$  and  $\mathfrak{m}_1$  invariant, and acts as a rotation by  $3\theta$  in the  $\mathfrak{m}_2$ -plane and in the  $\mathfrak{m}_3$ -plane. We use the standard realization of the 8-dimensional  $\text{Spin}(7)$ -representation  $\Delta_7$  as given in [BFGK91, p.97], and denote by  $\psi_i, i = 1, \dots, 8$  its basis. One then checks that  $\psi_3, \psi_4, \psi_5$  and  $\psi_6$  are fixed under the lift  $\tilde{\text{Ad}}(\theta)$  of the isotropy representation to  $\text{Spin}(7)$ . Thus, they define constant sections in the spinor bundle  $\Sigma N(1, 1) = SU(3) \times_{\tilde{\text{Ad}}} \Delta_7$ . The metric defined by  $s = 1, y = 2$  is exactly the 3-Sasakian metric on  $N(1, 1)$  (see the comments in the previous section); it has three Killing spinors with Killing number  $1/2$  ( $\psi_3, \psi_4, \psi_6$  in our notation). The metric defined by  $s = 1, y = 2/5$  is the Einstein metric with Killing spinor  $\psi_5$  and with Killing number  $-3/10$ , the well-known nearly parallel  $G_2$  structure on  $N(1, 1)$  (see [BFGK91, Thm 12, p.116]). It coincides with the nearly parallel  $G_2$  structure constructed by rescaling the underlying 3-Sasakian structure as described in Theorem 2.30.

In dimension 7, any connection  $\nabla$  with skew torsion  $H$  admitting a parallel spinor field defines a  $G_2$  structure of Fernandez-Gray type  $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$  on this manifold, and vice versa. This construction principle was used in [AF04] to define  $G_2$  structures on  $N(1, 1)$  via their parallel spinors. In particular, torsion forms  $T_i$  depending on the metric parameters  $s, y$  were given in Propositions 8.1–8.4 that admit  $\psi_i$  as parallel spinors,  $i = 3, \dots, 6$ . Let  $\nabla^i$  denote the connection with skew torsion  $T_i$ . We can summarize our results as follows:

**Theorem 2.31.** *On the Aloff-Wallach manifold  $N(1, 1)$  with the family of metrics  $g_{s,y}$ , the connection  $\nabla^i$ ,  $i = 3, \dots, 6$ , defines a  $G_2$  structure that is Einstein with skew torsion precisely for the values stated in the following table:*

Connection	Metric $g_{s,y}$	$s^\nabla/7$	Comment	Riemannian Einstein ?
$\nabla^3, \nabla^4$	$s = 1, y = 2$		3-Sasakian	yes
$\nabla^5$	$s = 1, y = \frac{2}{5}$		nearly parallel $G_2$	yes
$\nabla^5$	$s = 1, y = \frac{30}{13}$	$\frac{7245}{1352} \cong 5.3587$	new	no
$\nabla^6$	$s = 1, y = 2$		3-Sasakian	yes
$\nabla^6$	$s \cong 0.97833,$ $y \cong 0.34935$	$\cong 1.676989544$	new	no

Hence, we were able to construct two Einstein structures with skew torsion on  $N(1, 1)$  that go beyond the metrics predicted for theoretic reasons. It will be an interesting topic for further research to investigate their detailed geometrical properties.

*Remark 2.32.* In [AF04], one can also find a construction of 3-forms that make various linear combinations of the spinors  $\psi_3, \dots, \psi_6$  parallel. The algebraic systems of equations for the  $\nabla$ -Einstein condition are difficult to control, but tests with different parameters lead us to conjecture that for arbitrary  $a, b$  ( $ab \neq 0$ ) and  $a, b, c$  ( $abc \neq 0$ ) we obtain no new solutions.

In a same vein, one can ask for  $\nabla$ -Einstein structures for other embeddings of  $S^1$  into  $SU(3)$ , i.e. the general Aloff-Wallach manifold  $N(k, l)$ , with  $k, l \in \mathbb{N}$  coprime. For these, already the Riemannian Einstein metrics cannot be described explicitly, but only as solutions of a complicated system of equations depending on  $k, l$ , and the metric parameters  $s, y$ . Some tests for different value of  $k$  and  $l$  showed that further  $\nabla$ -Einstein structures do exist, but it seems hopeless to discuss the resulting system of equations in a reasonably general way for arbitrary  $k$  and  $l$ .

## A Appendix: Equivalent formulations of parallel torsion

Consider the one parameter family of connections with skew torsion given by

$$\nabla_X^s Y = \nabla_X^g Y + 2sH(X, Y), \quad s \in \mathbb{R}.$$

We will investigate the relation between parallel torsion and the vanishing of the first Bianchi identity. For the connection  $\nabla^s$  the following curvature identity holds

$$\begin{aligned} R^s(X, Y, Z, W) &= R^g(X, Y, Z, W) + 4s^2g(H(X, Y), H(Z, W)) + 4s^2\sigma_H(X, Y, Z, W) \\ &\quad + 2s\nabla_X^s H(Y, Z, W) - 2s\nabla_Y^s H(X, Z, W) \end{aligned}$$

and using the fact that

$$\nabla_X^s H(Y, Z, W) = \nabla_X^g H(Y, Z, W) - 2s\sigma_H(X, Y, Z, W)$$

it is easy to check that the first Bianchi identity reads as

$${}^{XYZ}_{\sigma} R^s(X, Y, Z, W) = -8s^2\sigma_H(X, Y, Z, W) + 4s(dH(X, Y, Z, W) + \nabla_W^g H(X, Y, Z)).$$

**Proposition A.1.** *For any real parameter  $s_0 \neq 0$ , the following conditions are equivalent:*

- (1)  $\nabla^{s_0} H = 0$  ;
- (2)  ${}^{XYZ}_{\sigma} R^{3s_0}(X, Y, Z, W) = 0$ ;
- (3)  $dH = 8s_0\sigma_H$ .

*Proof* — Suppose (1) holds, i.e. there exists  $s_0 \neq 0$  such that  $\nabla^{s_0} H = 0$ . Then  $\nabla^g H = 2s_0\sigma_H$ , which implies that  $dH = 8s_0\sigma_H$ . The Bianchi identity then reduces to

$${}^{XYZ}_{\sigma} R^s(X, Y, Z, W) = -8s(s - 3s_0)\sigma_H(X, Y, Z, W).$$

Hence, we have (2), the vanishing of the first Bianchi identity for  $s = 3s_0$ .

Consider now condition (2). The vanishing of the first Bianchi identity for some  $s \neq 0$  implies that

$$\nabla_W^g H(X, Y, Z) = 2s\sigma_H(X, Y, Z, W) - dH(X, Y, Z, W)$$

and from this expression we see that  $\nabla^g H$  is a totally antisymmetric tensor, and is therefore equal to  $\frac{1}{4}dH$ . Then

$$dH(X, Y, Z, W) + \frac{1}{4}dH(W, X, Y, Z) = 2s\sigma_H(X, Y, Z, W)$$

and simplifying we conclude that  $dH = \frac{8}{3}s\sigma_H$ , that is, (3) holds. But then the first equation can be rewritten  $\nabla^g H = \frac{2}{3}s\sigma_H$ , which is equivalent to  $\nabla^{s/3} H = 0$ , i.e. we proved (1).

It remains to deduce (1) from (3). For this, consider the general identity ([FI02], [Agr06, Cor. A.1.])

$$dH(X, Y, Z, W) = \frac{XYZ}{\sigma} [\nabla_X^s H(Y, Z, W)] - \nabla_W^s H(X, Y, Z) + 8s\sigma_H(X, Y, Z, W),$$

which holds under the assumption that  $s \neq 0$ . Under the assumption (3), it reduces to

$$\frac{XYZ}{\sigma} [\nabla_X^s H(Y, Z, W)] = \nabla_W^s H(X, Y, Z).$$

Both sides are tensorial quantities, but the left hand side is symmetric in  $X, Y$ , and  $Z$ , while the right hand side is antisymmetric in  $X, Y$ , and  $Z$ , hence they can only be equal if they vanish, which implies condition (1). □

Proposition A.1 is clearly wrong for the Levi-Civita connection ( $s_0 = 0$ ), showing the non-triviality of the result. In dimension 4, the 4-form  $\sigma_H$  vanishes for purely algebraic reasons; the theorem stays correct, indeed (1) and (2) hold then for any  $s_0 \in \mathbb{R}$ .

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